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BY
C.-G. ROSSBY

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A GENERALIZATION OF THE THEORY OF THE MIXING LENGTH WITH APPLICATIONS TO OCEANIC AND ATMOSPHERIC TURBULENCE

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I. INTRODUCTION

It has been shown by L. Prandtl[1] that the fictitious tangential stresses first introduced by Osborne Reynolds to account for the effect of superimposed turbulence on the mean motion of a fluid, in the case of straight, two-dimensional flow may be expressed in the form

$$\tau_x = \rho\, l^2 \frac{dU}{dz} \cdot \left| \frac{dU}{dz} \right|, \tag{1}$$

where ρ is the density and $\frac{dU}{dz}$ the rate of shearing of the mean motion, assumed to be directed along the x-axis. In the above formula, l, the "Mischungsweg" or molar free path, is a characteristic length of the state of turbulence and corresponds roughly to the mean free path in case of molecular motion. Prandtl and his collaborators[2] have succeeded to integrate the equations of motion in special cases, making use of (1). The integrals found agree well with observations.

The mixing length is not a constant, as in the molecular theory, but depends, in a fashion not yet fully known, upon the mean velocity distribution and upon the distance from and the character of the external boundaries. In the case of heterogeneous fluids moving in a gravitational field it is obvious that the value of l must depend also upon the stability of the medium under consideration.

Assuming in each point the existence of a certain fundamental turbulent disturbance restricted to the immediate vicinity of this point, v. Kármân[3] has been able to derive an independent proof for (1) by requiring that the patterns of these fundamental disturbances be kinematically similar, leaving only the length scale and the velocity scale in each disturbance undetermined. Furthermore, v. Kármân obtains an expression for l of the form

$$l = k \cdot \frac{\left| \frac{dU}{dz} \right|}{\left| \frac{d^2U}{dz^2} \right|} \tag{2}$$

[1] L. Prandtl: Bericht über Untersuchungen zur ausgebildeten Turbulenz, *Zeitschrift für angewandte Mathematik und Mechanik*, Band 5, 1925, p. 136.

[2] W. Tollmien: Berechnung turbulenter Ausbreitungsvorgänge, *Zeitschrift für angewandte Mathematik und Mechanik*, Band 6, 1926, p. 468.

[3] Th. v. Kármân: Mechanische Ähnlichkeit und Turbulenz, *Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen*, Mathematisch-Physikalische Klasse, 1930, Heft 1, p. 58.

in which k is a non-dimensional constant. From a combination of (1) and (2) it follows that the tangential stress may be calculated from the formula

$$\tau_x = \rho k^2 \cdot \frac{\left| \frac{dU}{dz} \right|^3 \frac{dU}{dz}}{\left(\frac{d^2 U}{dz^2} \right)^2} \tag{3}$$

With the aid of this expression for the stress v. Kármán has determined the theoretical velocity distribution in turbulent flow through circular pipes and derived a result which well agrees with the measured distribution. The constant k was found to have a value between 0.36 and 0.38.

In the study of large scale atmospheric and oceanic movements all but two of the fictitious tangential stresses introduced by Reynolds may be disregarded. Assuming the positive z-axis to be directed upward, the two significant stresses are

$$\tau_x = -\rho\, \overline{u'w'}, \quad \tau_y = -\rho\, \overline{v'w'}. \tag{4a}$$

The bar represents a mean, for instance the arithmetic mean over a certain unit time T, so that

$$\overline{f} = \frac{1}{T} \int_{t-\frac{1}{2}T}^{t+\frac{1}{2}T} f\, dt \tag{4b}$$

The dashes indicate deviations from the mean. Thus u', v', w' represent the components of the deviation of the instantaneous velocity vector from the mean vector in the point under consideration.

It is generally assumed that these deviations may be represented in (4a) by expressions of the type

$$u' = \pm \lambda_x \frac{dU}{dz}, \quad v' = \pm \lambda_y \frac{dV}{dz} \tag{5}$$

The positive sign is associated with negative w'-values and vice versa. U and V stand for the horizontal components of the mean motion. Thus the formulae for the stresses τ_x and τ_y reduce to

$$\tau_x = \rho\, \overline{\lambda_x w'} \cdot \frac{dU}{dz}, \quad \tau_y = \rho\, \overline{\lambda_y w'} \cdot \frac{dV}{dz}. \tag{6}$$

The coefficients $\rho\, \overline{\lambda_x w'}$ and $\rho\, \overline{\lambda_y w'}$ are normally, for lack of conclusive evidence to the contrary, assumed to be equal.[4] Under these conditions the fictitious turbulent stress per horizontal unit area is parallel to the shearing vector and may be written

$$\tau_x + i\, \tau_y = A \left[\frac{dU}{dz} + i \frac{dV}{dz} \right], \quad A = \rho\, \overline{\lambda w'}, \qquad (i = \sqrt{-1}) \tag{7}$$

[4] Certain data presented by L. F. Richardson in "Some Measurements of Atmospheric Turbulence," *Philosophical Transactions of the Royal Society of London*, Series A, Vol. 221, indicate a marked lack of isotropy in the eddy viscosity. Theoretical studies of the vertical distribution of the wind in the anisotropic case have been undertaken by S. Sakakibara in *The Geophysical Magazine*, Tokio, Vol. I, 130, 1930, and by Y. Isimaru in *Memoirs of the Imperial Marine Observatory*, Kobe, Japan, Vol. IV, No. 1, 1930. See also F. Möller: Austausch und Wind, *Meteorologische Zeitschrift*, Heft 2, 1931.

where A is the "Austausch" coefficient in Wilhelm Schmidt's terminology or "eddy viscosity" according to G. I. Taylor.[5] From the preceding discussion it is evident that A must depend upon the mean velocity distribution, upon stability and upon the distance from and character of the ground. It is assumed also that the same coefficient A enters into the expressions determining the eddy diffusion of heat, moisture and atmospheric or oceanic suspensions. Thus A is also a coefficient of "eddy-conductivity" and of "eddy-diffusivity."

The geophysical turbulence problem is really a twofold one. One phase concerns the determination of the diffusion of various atmospheric or oceanic properties when the state of turbulence (A) is known. This problem has been greatly advanced through investigations made by, among others, V. W. Ekman, G. I. Taylor, Hesselberg and Sverdrup and W. Schmidt.[6] The second phase of the turbulence problem consists in the development of a rational theory for the eddy-viscosity coefficient A.

In a general, qualitative way the variations of A with distance from the ground (ocean surface), with wind velocity (drift velocity) and with stability are known, but no theory has been presented which enables us to calculate A from these factors. The principal result of significance is a criterion developed by L. F. Richardson[7] which determines the conditions under which atmospheric turbulence will increase or decrease; however, this criterion is incomplete since it applies to volumes so large that diffusion of eddies through the boundaries may be neglected, and since it does not in a satisfactory way take into account the dissipation of eddy energy by molecular action. Even if these points should be remedied,[8] it is obvious that as long as the connection between eddy energy and the eddy-viscosity coefficient A remains obscure, Richardson's criterion will be of limited value in the determination of A.

It is evident that v. Kármán's theory, if at all applicable to atmospheric and oceanic phenomena, eliminates part of the difficulties enumerated above since it leads to an expression for the stress (3) which in a supposedly complete fashion takes into account the effect of the mean velocity distribution.

Through a comparison of (3) and (6) it is seen that

$$A = \rho k^2 \cdot \frac{\left|\dfrac{dU}{dz}\right|^3}{\left(\dfrac{d^2U}{dz^2}\right)^2} \tag{8}$$

In its original form v. Kármán's theory was restricted to the case of two-dimensional flow. Thus, with a view towards geophysical applications it becomes necessary to generalize the original theory so as to include motion of the atmospheric and oceanic type, that is, a mean motion in parallel horizontal planes, constant within each plane,

[5] G. I. Taylor: On Eddy-Motion in the Atmosphere, *Philosophical Transactions*, Series A, Vol. 215, p. 22.
W. Schmidt has summarized his numerous investigations in "Der Massenaustausch in freier Luft und verwandte Erscheinungen," *Probleme der Kosmischen Physik*, VII, Hamburg, 1925.

[6] In addition to the references given in fn. 5 see V. W. Ekman: On the Influence of the Earth's Rotation on Ocean Currents, *Arkiv f. Mat. Astr. o. Fys*, 2, Nr. 11, Stockholm, 1905, and Th. Hesselberg and H. U. Sverdrup: Die Reibung in der Atmosphäre, *Veröffentlichungen des Geophysikalischen Instituts der Universität Leipzig*, Heft 10, 1915.

[7] L. F. Richardson: The Supply of Energy from and to Atmospheric Eddies, *Proceedings of the Royal Society*, Series A, Vol. 97, No. A686.

[8] C.-G. Rossby: The Vertical Distribution of Atmospheric Eddy Energy, *Monthly Weather Review*, Vol. 54, No. 8.

but varying in azimuth and intensity from plane to plane. For the sake of convenience this particular type of motion will here be referred to as planar motion. The generalization will be carried out in section II.

It will be found that v. Kármán's condition of kinematic similarity can be satisfied only by a certain general type of planar motion. This permissible type is characterized by the fact that the endpoints of all the shearing vectors,

$$\frac{dU}{dz} + i\frac{dV}{dz},$$

form a logarithmic spiral in the *xy*-plane. We may refer to motion of this kind as logarithmic planar motion.

It is a well-established fact that the vertical distributions of the wind and of the drift in pure ocean drift currents belong to this general type. The result obtained would therefore seem to encourage the establishment of a theory of geophysical turbulence upon the basis created by v. Kármán.

It should be stressed that the justification for and the physical meaning of the condition of kinematic similarity remain somewhat obscure. This is emphasized by v. Kármán, who admits that the principal justification so far must be sought in the agreement between the theoretical results and observed facts. No exhaustive analysis of this question will be undertaken since it would require complete knowledge of the fundamental turbulent disturbance.

The generalization of v. Kármán's condition of similarity presented below is purely formal and is therefore neither more nor less acceptable than the original condition. It would seem then that even a partial agreement between conclusions drawn from this generalized theory and observed atmospheric and oceanic facts should add to the strength of the original theory.

In this paper no attempt will be made to calculate the effect of stability upon the molar free path.[9] In a homogeneous medium any fluid element may take the place of any other element; in a heterogeneous medium it must be assumed, either that each individual fluid element after passing through the fundamental turbulent disturbance returns to its original layer or else that mixing between strata of different densities takes place. If it be assumed, as is customary in meteorology and oceanography, that the mechanism which is responsible for the transport of momentum is responsible also for the turbulent diffusion of other properties, then the second of these alternatives must be accepted. Motion of this type, however, is not yet accessible to rigid mathematical treatment.

The expression derived by v. Kármán for the stress is not valid in the immediate vicinity of boundaries. The same restriction obviously applies to the generalized theory, which therefore must be supplemented with suitable, heuristic expressions for the stresses within the boundary layers.

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[9] This problem has recently been attacked by L. Prandtl in "Meteorologische Anwendung der Strömungslehre," *Beiträge zur Physik der freien Atmosphäre*, Bjerknes-Festband (XIX. Band), p. 188.

II. CONDITION OF KINEMATIC SIMILARITY

An incompressible, homogeneous liquid is maintained in a state of planar motion normal to the z-axis.

From the previously given definition of this type of motion, it follows that the velocity components U and V are functions of z only. By impressing suitable translations on the system, one may reduce the velocity at any given level to zero. Counting the z-coordinate from this particular level one has

$$U + iV = \left(U' + iV'\right) z + \frac{U'' + iV''}{2} \cdot z^2 + \ldots \tag{9}$$

The complex coefficients in this series may be written in plane polar coordinates,

$$U' + iV' = C' e^{i\psi'},\ U'' + iV'' = C'' e^{i\psi''},\ \ldots \tag{10}$$

Thus:

$$U + iV = C' e^{i\psi'} \cdot z + \frac{1}{2} C'' e^{i\psi''} \cdot z^2 + \ldots. \tag{11}$$

A variation in the orientation of the x-axis has the effect of changing the angles ψ', ψ'' etc., by a constant amount.

Now let a slight disturbance with the components $f_x\,(xyz)$, $f_y\,(xyz)$ and $f_z\,(xyz)$ be superimposed upon the mean motion defined in (11). The three components f_x, f_y, and f_z shall differ from zero only in the vicinity of $x = y = z = 0$, that is, the disturbance is limited to the immediate surroundings of this point. On account of the incompressibility of the fluid, the functions introduced above are not independent but connected mutually through the equation of continuity,

$$\frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} + \frac{\partial f_z}{\partial z} = 0. \tag{12}$$

The total flow is given by the two expressions,

$$U_t + iV_t = C' e^{i\psi'} z + \frac{1}{2} C'' e^{i\psi''} z^2 + \ldots + f_x + i f_y \tag{13}$$

and

$$W_t = f_z. \tag{14}$$

The subscript t stands for total (velocity).

It is assumed that the disturbance defined above remains stationary, at least for a short period of time. During that time it is carried along bodily with the mean motion of the fluid. However, it may be shown that this assumption is not essential to the following argument. Let such a pattern of flow exist at every point.

We shall now require that the patterns of flow defined by (13) and (14) be kinematically similar with a length and velocity scale that may vary from point to point. Since the mean motion varies in azimuth from one level to the next, it is evident that we must permit a variation in azimuth of our coordinate system in order to bring out the kinematic similarity.

It is possible to determine by simple methods the requirements for this kinematic similarity.

We compare conditions at two levels, L_1 and L_2. In the vicinity of two points P_1 and P_2, one at each level, the small disturbances D_1 and D_2 are superimposed upon the mean motion. D_1 differs from zero only in the immediate vicinity of P_1 and D_2 is zero everywhere except in the neighborhood of P_2. If the patterns of flow in the vicinity of these two levels are to be kinematically similar when referred to P_1 and P_2, *it follows that the mean motions at these two levels must be kinematically similar.*

We introduce a coordinate system $(x\ y\ z)$ with its origin in P_2 and another system $(\xi \eta \zeta)$ with its origin at P_1. If the length scale of the two patterns be l and the velocity scale be $\frac{A}{l}$, we have, *for corresponding points,*

$$x = l\xi,\ y = l\eta,\ z = l\zeta. \tag{15}$$

Furthermore,

$$f_x = \frac{A}{l} f_\xi,\ f_y = \frac{A}{l} f_\eta,\ f_z = \frac{A}{l} f_\zeta. \tag{16}$$

The symbol A, introduced in conformity with v. Kármân's notation, should not be confused with the "Austausch"-coefficient A.

The mean velocity in the vicinity of P_2 is given by $U_2 + iV_2$, where U_2 and V_2 are the components along the x- and y-axes. The *same* velocity, resolved at P_2 along the ξ- and η-directions is $U_{2_1} + iV_{2_1}$. Then

$$U_2 + i\,V_2 = \left(U_{2_1} + i\,V_{2_1}\right) e^{-i\varphi}, \tag{17}$$

φ being the angle which the x-axis forms with the ξ-axis (counted positive from ξ toward η).

If

$$U_{2_1} + i\,V_{2_1} = C_2'\, e^{i\psi_2'} \cdot z + \frac{1}{2} C_2''\, e^{i\psi_2''} \cdot z^2 + \dots \tag{18}$$

we have

$$U_2 + i\,V_2 = C_2'\, e^{i(\psi_2' - \varphi)}\, z + \frac{1}{2} C_2''\, e^{i(\psi_2'' - \varphi)}\, z^2 + \dots \tag{19}$$

In P_1 the mean motion is given by

$$U_1 + i\,V_1 = C_1'\, e^{i\psi_1'}\, \zeta + \frac{1}{2} C_1''\, e^{i\psi_1''} \cdot \zeta^2 + \dots . \tag{20}$$

Thus the angles ψ_2', ψ_1', ψ_2'', ψ_1'' etc., are now measured from the same fixed direction.

In view of the required kinematic similarity of the two patterns when referred to the xyz-system and the $\xi\eta\zeta$-system respectively, we must have, in each corresponding pair of points,

$$\boxed{U_2 + i\,V_2 = \frac{A}{l}\left(U_1 + i\,V_1\right)} \tag{21}$$

It follows that

(22a) $$\frac{d}{dz}\left(U_2 + i V_2\right)_{z=0} = \frac{A}{l^2}\frac{d}{d\zeta}\left(U_1 + i V_1\right)_{\zeta=0}.$$

(22b) $$\frac{d^2}{dz^2}\left(U_2 + i V_2\right)_{z=0} = \frac{A}{l^3}\frac{d^2}{d\zeta^2}\left(U_1 + i V_1\right)_{\zeta=0}$$

etc.

Through (15) and (16) we have insured kinematic similarity of the two disturbances. Through (22a, b, etc.) we have insured kinematic similarity of the mean motion at the two levels, the length and velocity scales being the same as those of the disturbances. Thus it is obvious that the total motion will be kinematically similar in the two points. This may be verified through direct substitution in Helmholtz' equations for the transfer of vorticity.[10]

If we introduce the values for $U_2 + i V_2$ and $U_1 + i V_1$ from (19) and (20) we find, from (22a),

(23) $$C_2' \, e^{i(\psi_2' - \varphi)} = \frac{A}{l^2} C_1' \, e^{i\psi_1'}$$

or

I. $$\boxed{\frac{A}{l^2} = \frac{C_2'}{C_1'},}$$

and

II. $$\boxed{\varphi = \psi_2' - \psi_1'}$$

Since the orientation of the ξ-axis is arbitrary, it is permissible to assume that $\psi_1' = 0$. Thus, going from one level to another, we find that the patterns of flow are rotated through an angle which is equal to the rotation of the shearing vector, $\frac{d}{dz}(U + iV)$.

It is now possible to form an expression for the tangential stress in the vicinity of P_2. Its components, resolved along the x- and y-axes, are τ_{x2} and τ_{y2}. Resolving the same stress with respect to the fixed directions ξ and η, one obtains

(24) $$\tau_{\xi 2} + i \tau_{\eta 2} = e^{i\varphi} \cdot \left(\tau_{x2} + i \tau_{y2}\right).$$

But, according to (4a),

(25) $$\tau_{x2} + i\tau_{y2} = -\rho\left(\overline{f_x f_z} + i\,\overline{f_y f_z}\right)$$

$$= -\rho\frac{A^2}{l^2}\left(\overline{f_\xi f_\zeta} + i\,\overline{f_\eta f_\zeta}\right) = \frac{A^2}{l^2}\left(\tau_{\xi 1} + i\,\tau_{\eta 1}\right)$$

[10] Horace Lamb: Hydrodynamics, fourth edition, *Cambridge University Press*, 1916, p. 198.

Thus, making use of I and II, we find

III.
$$\boxed{\tau_x + i\tau_y = \rho l^2 C'^2 \cdot e^{i(\psi' + \sigma)}}$$

In the above formula unnecessary subscripts have been dropped. σ is a constant angle, which may or may not be equal to zero. A constant factor has been absorbed in l, which as yet remains undetermined. The above expression indicates that the stress forms a constant angle σ with the direction of the shearing vector. The stress is proportional to the square of the rate of shearing (C') and to the square of the scale factor l, the mixing length. If σ is zero and the discussion be applied to two-dimensional motion, the above formula is identical with the one derived by Prandtl.

If we now, in formulating our condition of kinematic similarity, take into account also terms of second order in z in the mean motion, we find, from (19), (20), and (22b), the following relation:

(26)
$$C_2'' e^{i(\psi_2'' - \varphi)} = \frac{A}{l^3} C_1'' e^{i\psi_1''}$$

Thus,

(27)
$$\frac{A}{l^3} = \frac{C_2''}{C_1''},$$

or, if we make use of I,

IV.
$$\boxed{l = k\frac{C'}{C''}},$$

where k is a proportionality factor. This expression, identical with the one obtained by v. Kármán, determines the mixing length in terms of the mean velocity distribution.

From (26) and II follows that

(28)
$$\varphi = \psi_2'' - \psi_1'' = \psi_2' - \psi_1'$$

or

V.
$$\boxed{\psi'' - \psi' = \text{constant.}}$$

Thus a new condition has now been impressed upon the mean motion, restricting the permissible planar motions to the particular type in which the shearing vector and its rate of change normal to the plane of motion always form a constant angle. This condition may also be formulated in another way. From the definitions

(29)
$$\frac{d^2}{dz^2}(U + iV) = C'' e^{i\psi''}$$

and

$$\frac{d}{dz}\left(U+iV\right)=C'e^{i\psi'} \tag{30}$$

we obtain

$$\frac{d}{dz}\left(C'e^{i\psi'}\right)=\left(\frac{dC'}{dz}+iC'\frac{d\psi'}{dz}\right)e^{i\psi'}=C''e^{i\psi''} \tag{31}$$

If we introduce a constant α defined by

$$\tan\left(\psi''-\psi'\right)=\frac{1}{\alpha}, \tag{32}$$

we find, from (31) and (32),

$$C'\frac{d\psi'}{dz}=\frac{1}{\alpha}\frac{dC'}{dz} \tag{33}$$

or, after integration,

$$C'=Ke^{\alpha\psi'} \tag{34}$$

Thus, if projected on one plane and set off from the same point with ψ' as azimuth, the endpoints of the shearing vectors form a logarithmic spiral. This particular type of planar motion shall be referred to as logarithmic planar motion. In the theory for pure drift currents developed by Ekman, planar motion of this particular type is obtained. In Ekman's theory the angle $\psi''-\psi'$ has the value $\frac{\pi}{4}$.

Making use of (34) and IV we can write the expression for the stress in a different form. From (33) and (31) it follows that

$$\left(C''\right)^2=\left(\frac{dC'}{dz}\right)^2+C'^2\left(\frac{d\psi'}{dz}\right)^2=C'^2\left(\frac{d\psi'}{dz}\right)^2\left(1+\alpha^2\right). \tag{35}$$

Thus

$$\frac{C'^4}{C''^2}=\frac{C'^2}{1+\alpha^2}\left(\frac{dz}{d\psi'}\right)^2=\frac{K^2}{1+\alpha^2}\left(\frac{dz}{d\psi'}\right)^2e^{2\alpha\psi'} \tag{36}$$

and, finally,

$$\text{VI.}\qquad \boxed{\tau_x+i\tau_y=\frac{\rho k^2K^2}{1+\alpha^2}\left(\frac{dz}{d\psi}\right)^2e^{2\alpha\psi+i(\psi+\sigma)}}$$

In VI we have dropped some unnecessary dashes and subscripts. It appears that the mean motion is now determined with exception for the rate at which the shearing vector rotates with elevation, $\frac{d\psi}{dz}$. A comparison of (34), (35) and IV shows that the mixing length is inversely proportional to the rate of rotation of the shearing vector.

The preceding discussion well illustrates the arbitrary character of the condition of kinematic similarity. This condition implies that the mean motion in the vicinity of two different levels must be kinematically similar. If we consider terms of the first order only, one of our scale factors, l, remains undetermined and we are led to Prandtl's expression for the stress. By considering terms of the second order in z, one obtains a formula for l in terms of the mean velocity distribution, but at the same time a certain restriction is impressed upon the mean motion. If terms of higher order are considered, additional restrictions upon the mean motion are introduced and cannot generally be fulfilled. Thus it would seem that the success of the method outlined above essentially depends upon the fact that the linear dimensions of the turbulent disturbances are so small that within their range the mean motion may be sufficiently accurately described by means of the two first terms in the development (9).

From the equations (4a), (5) and (6) it is seen that the constant k may be said to measure the correlation between u' (or v') and w'. When it is considered that the significant turbulent velocity fluctuations in pipes have a frequency of several hundred per second while the corresponding frequency in the atmosphere or in the ocean is in the vicinity of 15 per minute,[11] it is reasonable to expect that the correlation between u' and w' must be decidedly smaller in geophysical than in laboratory turbulence. The applications discussed below show that this is the case.

The circumstance that the fundamental turbulent disturbance in our case is three-dimensional and not two-dimensional as in the problem discussed by v. Kármân may also contribute to the explanation of the discrepancies between values of k obtained from laboratory experiments and from geophysical measurements. We may express this by saying that the value of k occurring in (VI) is a function of α.

[11] Sir Napier Shaw: Manual of Meteorology, Volume IV (Revised edition of Part IV), *Cambridge University Press*, 1931, p. 144.

III. RESTRICTION IMPOSED UPON THE MEAN MOTION DUE TO THE CORIOLIS' FORCE OF THE EARTH'S ROTATION

In the preceding section we have derived an expression for the tangential stress of the form

$$\tau_x + i\tau_y = \frac{\rho k^2 K^2}{1+\alpha^2}\left(\frac{dz}{d\psi}\right)^2 e^{2\alpha\psi + i(\psi+\sigma)} \tag{37}$$

When the above formula is applied to the study of geophysical phenomena it will be found that the deflecting force of the earth's rotation places an additional restriction on the types of logarithmic planar motion possible in the ocean or in the atmosphere. It is the purpose of this section to derive this restriction.

The horizontal deflecting force per unit mass $(D_x + iD_y)$ of the earth's rotation is proportional to the velocity relative to the surface of the earth and, on the northern hemisphere, directed ninety degrees to the right of the velocity vector. Thus

$$D_x + iD_y = f e^{-\frac{\pi}{2}i}\left(U + iV\right), \qquad f = 2\,\Omega \sin L \tag{38}$$

Here Ω represents the angular velocity of the earth and L the latitude (positive on the northern hemisphere, negative on the southern hemisphere).

If the motion takes place under the influence of a *vertically and horizontally constant* horizontal pressure gradient $\frac{\partial p}{\partial x} + i\frac{\partial p}{\partial y}$, we have

$$0 = \rho f \cdot e^{-\frac{\pi}{2}i}\left(U+iV\right) - \left(\frac{\partial p}{\partial x} + i\frac{\partial p}{\partial y}\right) + \frac{\rho k^2 K^2}{1+\alpha^2}\frac{d}{dz}\left[\left(\frac{dz}{d\psi}\right)^2 e^{2\alpha\psi + i(\psi+\sigma)}\right]. \tag{39}$$

Differentiating this equation once with respect to z and making use of the fact that

$$\frac{d}{dz}\left(U + iV\right) = K e^{\alpha\psi + i\psi}, \tag{40}$$

we find

$$0 = \rho f K e^{\alpha\psi + i(\psi - \frac{\pi}{2})} + \frac{\rho k^2 K^2}{1+\alpha^2}\frac{d^2}{dz^2}\left[\left(\frac{dz}{d\psi}\right)^2 e^{2\alpha\psi + i(\psi+\sigma)}\right] \tag{41}$$

or

$$0 = \frac{f(1+\alpha^2)}{k^2 K} e^{\alpha\psi + i(\psi - \frac{\pi}{2} - \sigma)} + \frac{d^2}{dz^2}\left[\left(\frac{dz}{d\psi}\right)^2 e^{2\alpha\psi + i\psi}\right]. \tag{42}$$

The double differentiation indicated in the second term may easily be carried out and gives the result

$$\frac{d^2}{dz^2}\left[\left(\frac{dz}{d\psi}\right)^2 e^{(2\alpha+i)\psi}\right] = \frac{2z''' + 3\,(2\alpha+i)z'' + (2\alpha+i)^2 z'}{z'} e^{(2\alpha+i)\psi}, \tag{43}$$

in which expression

$$z' = \frac{dz}{d\psi},\ z'' = \frac{d^2z}{d\psi^2},\ z''' = \frac{d^3z}{d\psi^3}. \tag{44}$$

Thus we find, from (42) and (43),

$$0 = \frac{f(1+\alpha^2)}{k^2K} e^{-i(\frac{\pi}{2}+\sigma)+(\alpha+i)\psi} \cdot z' + \left[2z''' + 3(2\alpha+i)z'' + (2\alpha+i)^2 z'\right] e^{(2\alpha+i)\psi}. \tag{45}$$

If we introduce the abbreviation

$$M + iN = e^{-i(\frac{\pi}{2}+\sigma)} \cdot \frac{f(1+\alpha^2)}{k^2K}, \qquad (M \text{ and } N \text{ real}), \tag{46}$$

the complex equation (45) reduces to

$$2z''' + 3(2\alpha+i)z'' + (2\alpha+i)^2 z' + (M+iN) z' e^{-\alpha\psi} = 0. \tag{47}$$

This is a linear, second order differential equation for z'. Since z' necessarily must be real it is obvious that the preceding equation actually consists of two which must be satisfied simultaneously. This is not possible generally but may be achieved if the parameters have certain definite values.

The two equations are

$$2z''' + 6\alpha z'' + \left[(4\alpha^2-1) + Me^{-\alpha\psi}\right] z' = 0. \tag{48}$$

$$3z'' + \left[4\alpha + Ne^{-\alpha\psi}\right] z' = 0. \tag{49}$$

By successive differentiations of the second of these two equations and substitution in the first, one obtains an expression of the form

$$t_1 e^{-2\alpha\psi} + t_2 e^{-\alpha\psi} + t_3 = 0 \tag{50}$$

which must be *identically* satisfied.

In (50) the coefficients t_1 and t_2 and t_3 depend upon the three parameters M, N, and α only. Assuming at first that $\alpha \neq 0$ it is evident that equation (50) can be satisfied identically then and then only, when

$$t_1 = t_2 = t_3 = 0 \qquad (\alpha \neq 0) \tag{51}$$

But t_1 is equal to $\frac{2N^2}{9}$. Thus t_1 will disappear only if N disappears. According to (46) this would occur when

$$\frac{\pi}{2} + \sigma = \pm n\pi \qquad (n \text{ integer}) \tag{52}$$

Thus

$$\sigma = -\frac{\pi}{2} \pm n\pi \tag{53}$$

which would imply that the stress everywhere is directed at right angles to the shearing vector. If this alternative be discarded as contrary to available evidence, our only way of satisfying (50) identically is to require

$$\alpha = 0 \tag{54}$$

and

$$\left(t_1 + t_2 + t_3\right)_{\alpha=0} = 0. \tag{55}$$

The second of these relations shows that one of the two parameters M and N may be expressed in terms of the other. We shall now inquire into the meaning of the condition $\alpha = 0$.

Generally, in the case of logarithmic planar motion, one has

$$\frac{dU}{dz} + i\,\frac{dV}{dz} = K\,e^{\,(\alpha+i)\psi} \tag{56}$$

and, therefore, in accordance with (54),

$$\frac{dU}{dz} + i\,\frac{dV}{dz} = Ke^{i\psi}. \tag{57}$$

Thus the shearing vector has everywhere the same absolute value but changes direction from level to level. The endpoints of this vector now form a circle. Since

$$\tan\left(\psi'' - \psi'\right) = \frac{1}{\alpha} \tag{32}$$

it is evident that

$$\psi'' = \psi' \pm \frac{\pi}{2} \tag{58}$$

and thus *the shearing vector and its vectorial rate of change with elevation must be directed at right angles to each other.*

It has been pointed out previously that the corresponding angle in Ekman's theory is $\frac{\pi}{4}$, or only one-half of the value derived from v. Kármán's theory.

The stress reduces to the form

$$\boxed{\tau_x + i\,\tau_y = \rho k^2 K^2 \left(\frac{dz}{d\psi}\right)^2 e^{i(\psi+\sigma)}.} \tag{VII}$$

IV. APPLICATION TO PURE DRIFT CURRENTS

A wind of uniform velocity is blowing over a very large ocean surface and exerts a stress τ_0 per unit area. The nature of the resulting drift current is sought. The water is homogeneous and its surface is level. Thus there are no horizontal pressure gradients to consider.

In this case it is advantageous to place the origin in the ocean surface with the z-axis directed vertically downward. The x-axis points in the direction of the force exerted by the wind and the y-axis is directed 90° to the right from the x-axis. Then

$$\tau_x + i\tau_y = \rho k^2 K^2 \left(\frac{dz}{d\psi}\right)^2 e^{i(\psi+\sigma)} \tag{59}$$

represents the stress exerted at the depth z by the water below this level on the water above.

In view of the particular orientation of the coordinate system, the deflecting force has the value

$$D_x + iD_y = \rho f e^{\frac{\pi}{2}i} (u + iv). \tag{60}$$

In this formula $u + iv$ represents the drift velocity vector.

Since there are no horizontal pressure gradients to consider, the equations of motion may be condensed into one,

$$\rho f e^{\frac{\pi}{2}i} (u + iv) + \rho k^2 K^2 \frac{d}{dz}\left[\left(\frac{dz}{d\psi}\right)^2 e^{i(\psi+\sigma)}\right] = 0. \tag{61}$$

Differentiation gives

$$\frac{f}{k^2 K} e^{(\frac{\pi}{2}-\sigma)i+i\psi} + \frac{d^2}{dz^2}\left[\left(\frac{dz}{d\psi}\right)^2 e^{i\psi}\right] = 0, \tag{62}$$

or, after completion of the differentiation in the second term,

$$2z''' + 3iz'' - z' + \frac{f}{k^2 K} e^{(\frac{\pi}{2}-\sigma)i} \cdot z' = 0. \tag{63}$$

We must therefore have, simultaneously,

$$2z''' = z' [1 - q \sin \sigma] \tag{64}$$

$$3z'' = -q \cos \sigma \cdot z' \tag{65}$$

$$q = \frac{f}{k^2 K} \tag{66}$$

These equations can be fulfilled only if

$$\frac{q^2 \cos^2 \sigma}{9} = \frac{1}{2}(1 - q \sin \sigma) \tag{67}$$

Then

$$z = R \cdot e^{-\frac{q \cos \sigma \cdot \psi}{3}} + S, \tag{68}$$

where R and S are constants of integration. We know from experience that ψ increases with z. When ψ approaches infinity, z approaches a certain limiting depth h,

$$h = S \tag{69}$$

There is also a solution corresponding to a ψ which decreases with depth, but this solution must be discarded since it leads to values of the drift velocity which increase downward.

At sea level, ψ has the value ψ_o. Thus

$$z = h\left[1 - e^{\frac{q \cos \sigma}{3}(\psi_o - \psi)}\right]. \tag{70}$$

The drift velocity at any level is obtained from

$$u + iv = u_s + iv_s + \int_0^z \frac{d}{dz}(u + iv)\, dz, \tag{71}$$

in which formula $u_s + iv_s$ represents the surface drift. We have

$$\frac{d}{dz}(u + iv) = K e^{i\psi} \tag{72}$$

and from (70),

$$\frac{dz}{d\psi} = h \frac{q \cos \sigma}{3} e^{\frac{q \cos \sigma}{3}(\psi_o - \psi)}. \tag{73}$$

Making use of (72) and (73) we find

$$u + iv = u_s + iv_s + h \cdot \frac{q \cos \sigma}{3} \cdot K \int_{\psi_o}^{\psi} e^{i\psi + \frac{q \cos \sigma}{3}(\psi_o - \psi)}\, d\psi. \tag{74}$$

The integration indicated in (74) is easily performed and yields the result

$$u + iv = u_s + iv_s + h\,K \cos \beta \cdot e^{i\beta}\left[e^{i\psi_o} - e^{i\psi + \operatorname{cotg} \beta\,(\psi_o - \psi)}\right], \tag{75}$$

in which β is defined by

$$\operatorname{cotg} \beta = \frac{q \cos \sigma}{3}. \tag{76}$$

At the depth h the drift velocity vanishes. Thus

$$u_s + iv_s = -hK \cos \beta \cdot e^{i(\psi_o + \beta)}, \tag{77}$$

and therefore, finally

$$u + iv = (u_s + iv_s)\, e^{(\operatorname{cotg} \beta - i)(\psi_o - \psi)}. \tag{78}$$

Thus the drift velocity vector itself forms a logarithmic spiral.

We must now determine the surface drift $u_s + i v_s$. The stress at the depth z is, according to (59) and (73), given by

$$\tau_x + i\tau_y = \rho k^2 K^2 h^2 \operatorname{cotg}^2 \beta \cdot e^{2 \operatorname{cotg} \beta\, (\psi_o - \psi) + i\, (\psi + \sigma)} \tag{79}$$

At the surface, $z = 0$, this value reduces to

$$(\tau_x + i\tau_y)_{z=0} = \rho k^2 K^2 h^2 \operatorname{cotg}^2 \beta \cdot e^{i\, (\psi_o + \sigma)}. \tag{80}$$

This force must be equal and opposite to the wind force τ_o. Thus

$$\pi = \psi_o + \sigma \tag{81a}$$

$$\tau_o = \rho k^2 K^2 h^2 \operatorname{cotg}^2 \beta. \tag{81b}$$

Inserting (81a) in (77) we find, for the surface drift,

$$u_s + i v_s = h\, K \cos \beta\, e^{i\, (\beta - \sigma)} \tag{82}$$

Thus

$$w_s = \sqrt{u_s^2 + v_s^2} = h\, K \cos \beta \tag{83}$$

and

$$\tau_o = \frac{\rho k^2 w_s^2}{\sin^2 \beta}. \tag{84}$$

The unknown quantities σ, β and q are connected through the two equations

$$\frac{q^2 \cos^2 \sigma}{9} = \frac{1}{2}\,(1 - q \sin \sigma) \tag{67}$$

$$\frac{q \cos \sigma}{3} = \operatorname{cotg} \beta. \tag{76}$$

In addition we have, by definition,

$$q = \frac{f}{k^2 K}. \tag{85}$$

If we know the angle σ which the stress forms with the shearing vector, the values of q and thus also of K follow from (67) and (85). By substitution in (81b) we may then determine h, the depth of the drift current, as a function of the surface stress.

We assume at first that the stress is parallel to the vector of shear, or

$$\sigma = 0. \tag{86}$$

It then follows from (67) that

$$q = \frac{3}{\sqrt{2}} \tag{87}$$

and from (85) that

(88) $$K = \frac{f\sqrt{2}}{3k^2}.$$

From (76) we find

(89) $$\tan\beta = \sqrt{2}, \quad \beta = 54^\circ\,44'$$

It follows from (82) that the surface drift is directed $54^\circ\,44'$ to the right of the wind force. We may calculate the surface drift velocity from (84), making use of the β—value given in (89).

(90) $$\tau_o = \frac{3}{2}\,\rho\,k^2\,w_s^2.$$

To determine the depth of the drift current we employ (83) which gives us

(91) $$w_s = \frac{hK}{\sqrt{3}}$$

or, with the aid of (88),

(92) $$h = \frac{3\sqrt{3}}{\sqrt{2}}\,\frac{k^2}{f}\,w_s = \frac{3k}{f}\sqrt{\frac{\tau_o}{\rho}}, \quad \tau_o = \rho\,\frac{f^2h^2}{9k^2}.$$

The formulae (88), (89), (90) and (92) completely determine the drift current, assuming $\sigma = 0$. In order to evaluate these expressions numerically it is, of course, necessary to know the constant k. In v. Kármán's original, two-dimensional theory $k = 0.38$, but it has already been pointed out that we must not expect the same value to hold also in the case of planar motion.

One objection to the solution just given may be based on the fact that it leads to a deflection of the surface current relative to the direction of the wind stress of $54^\circ\,44'$. Assuming surface wind and surface stress to be parallel, this angle seems somewhat too large. Ekman's theory leads to the value $\frac{\pi}{4}$, which apparently agrees better with observations, although P. H. Gallé[12] has found, from observations in the region between 10°–20° southerly latitude, an average deflection of 49°.

If we discard the assumption that the stress is everywhere directed along the shearing vector and assume instead that there is a constant angle between the two, then σ may be so determined as to give a surface drift deflection of 45°. The calculation is carried out below.

Going back to the expression (82) for the surface drift vector, we now require

(93). $$\beta - \sigma = \frac{\pi}{4}.$$

To determine β and q one may introduce the symbols

(94) $$x = q\cos\sigma, \quad y = q\sin\sigma.$$

[12] See O. Krümmel: Handbuch der Ozeanographie, Band II, Stuttgart, 1911, p. 456.

Thus (67) transforms into

$$x^2 = \frac{9}{2}(1 - y) \tag{95}$$

and (76) may be written

$$x = 3 \operatorname{cotg} \beta \tag{96}$$

From (93) we find

$$\operatorname{cotg} \beta = \frac{x - y}{x + y} \tag{97}$$

and thus, combining (96) and (97),

$$\frac{x}{3} = \frac{x - y}{x + y} \quad \text{or} \quad \frac{x(3 - x)}{3 + x} = y. \tag{98}$$

The value of y obtained in (98) may be inserted in (95). This gives, after reductions,

$$(2x - 3)(x^2 + 9) = 0. \tag{99}$$

The only real solution is

$$x = \frac{3}{2}, y = \frac{1}{2} \tag{100}$$

or

$$\tan \sigma = \frac{1}{3}, \tan \beta = 2. \tag{101}$$

It follows that

$$\sigma = 18^\circ 26', \beta = 63^\circ 26', q = \frac{1}{2}\sqrt{10}. \tag{102}$$

Thus

$$K = \frac{2f}{k^2\sqrt{10}}, \quad \tau_o = \frac{\rho}{10}\frac{f^2 h^2}{k^2} = \frac{5}{4}\rho k^2 w_s^2. \tag{103}$$

The two alternative solutions derived above are reproduced graphically in Fig. 1. The spirals represent the endpoints of all the vectors $u + iv$ projected on the plane of the paper and set off from one point. It should be remembered that w_s (the surface drift velocity) and h have different values in the two solutions, even though the external parameters τ_o, ρ and f are the same. Indicating the values corresponding to $\sigma = 0$ by a dash, we find the ratios

$$\frac{w_s}{w_s'} = \sqrt{\frac{6}{5}}, \quad \frac{h}{h'} = \sqrt{\frac{10}{9}}, \tag{104}$$

provided k is the same in both solutions.

In the diagram the scale has been chosen in such a fashion that $w_s' = 1$. Thus all velocities obtained from the spirals are expressed in fractions of the corresponding surface drift for $\sigma = 0$. In order to compare the flow at different levels, we have set off, on the curve $\sigma = 0$, the points corresponding to $\frac{z}{h'} = 0.1, 0.2, 0.3$, etc. On the other curve we have marked the points

(105) $$\frac{z}{h} = 0.1\sqrt{\frac{9}{10}},\quad 0.2\sqrt{\frac{9}{10}},\quad 0.3\sqrt{\frac{9}{10}}, \text{ etc.}$$

so that corresponding points on the two curves refer to the same absolute rather than to the same relative depth.

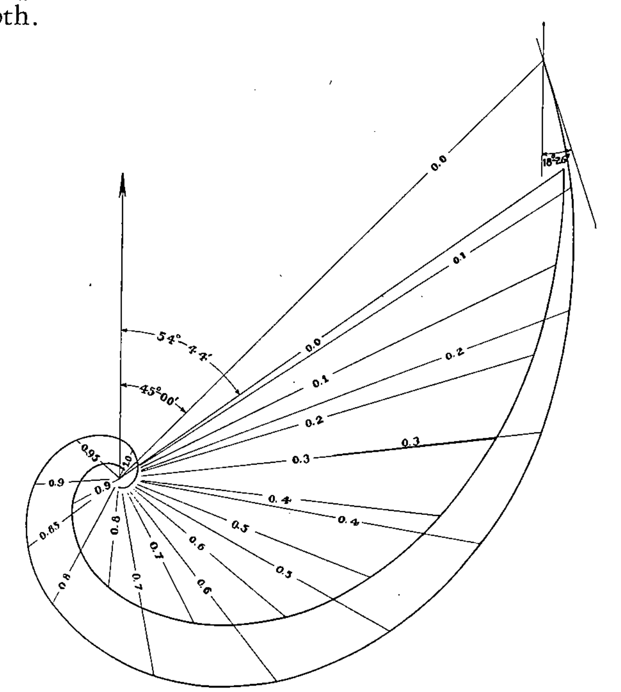

FIG. 1

The first spiral ($\sigma = 0$) is, according to (68), (78) and (89), given in parameter form by

(106a) $$\frac{w'}{w_s'} e^{i(\varphi-\varphi_s')} = e^{\left(\frac{1}{\sqrt{2}}-i\right)(\psi_0-\psi)} \qquad (\varphi_s' = 54^\circ\, 44')$$

(106b) $$\frac{z}{h'} = 1 - e^{\frac{\psi_0-\psi}{\sqrt{2}}}$$

In this system $\frac{w'}{w'_s}$ represents the drift velocity at the depth z in percentage of the surface drift. The angles φ' and φ_s' indicate the relative orientation of these two drifts.

The second spiral is given by a similar system, obtained from (68), (78) and (101).

(107a) $$\frac{w}{w_s} e^{i(\varphi-\varphi_s)} = e^{\left(\frac{1}{2}-i\right)(\psi_0-\psi)} \qquad [\varphi_s = 45^\circ]$$

(107b) $$\frac{z}{h} = 1 - e^{\frac{\psi_0-\psi}{2}}$$

As mentioned before, it is desirable to compare velocities at the same absolute depths. For this purpose one may transform (107a) into

$$\frac{w}{w_s'} e^{i(\varphi-\varphi_s)} = \sqrt{\frac{6}{5}}\, e^{(\frac{1}{2}-i)(\psi_o-\psi)} \tag{108a}$$

and (107b) into

$$\frac{z}{h'} = \sqrt{\frac{10}{9}} \left[1 - e^{\frac{\psi_o-\psi}{2}}\right] \tag{108b}$$

The drift currents analyzed in this paper differ from Ekman's in three particular respects.[13] In the first place, according to the present theory the quadratic relation between stress and surface drift is independent of latitude, whereas Ekman finds that the surface drift produced by a given stress must be inversely proportional to the square root of the sine for the latitude. This would mean that the surface drift corresponding to a given stress reaches very high values in the vicinity of the equator. Since the stress is never measured directly, the customary way of verifying this particular result is to compare drift velocity and surface wind velocity and to assume a definite relation between stress and surface wind. The particular empirical relation employed by Ekman for the latter purpose assumes that the stress is independent of the latitude but proportional to the square of the wind velocity W_o,

$$\tau_o = 32 \cdot 10^{-7} \cdot W_o^2 . \tag{109}$$

The second important deviation from Ekman's theory is to be found in the relation between the surface drift velocity and the depth of the drift current h. This relation is here given by (83) and (85) which, if combined, give

$$h = \frac{k^2 q}{\cos\beta} \cdot \frac{w_s}{f} . \tag{110}$$

Thus the depth of the drift current is inversely proportional to the sine of the latitude and proportional to the surface drift. Ekman's corresponding, semi-empirical formula, takes the form

$$D = 600\, w_s , \tag{111}$$

where D is the "depth of frictional influence" and fairly closely corresponds to h. Observations indicate a pronounced increase in the depth h (or D) towards the equator[14], but this may perhaps partly be ascribed to the permanent character of the trade winds as compared with winds of higher latitudes.

A third and actually the most fundamental difference between the preceding theory and that of Ekman may be seen in the fact that the latter introduces four external parameters τ, ρ, f and A (eddy-viscosity) or D (depth of frictional influence). When these four quantities are known, the drift current may be calculated in all its details. It was brought out in the introduction to this paper that A, the eddy-viscosity coefficient, is not a constant but varies with depth. For the sake of convenience, this fact is disregarded

[13] Recently a good summary of Ekman's theory has been presented by G. F. McEwen in *Bulletin of the National Research Council*, Number 85, Oceanography, Chapter 8, Washington, 1932.

[14] A. Defant: Dynamische Ozeanographie, *Einführung in die Geophysik*, III, Julius Springer, Berlin, 1929, p. 74.

in Ekman's original theory[15] and a mean value of A is introduced. The mean value obviously must vary with the stability of the stratification in some fashion which neither theory at present is capable of indicating, but if the discussion be restricted to the case of homogeneous water, there are only three independent external parameters, τ, ρ and f. It is the particular advantage of the theory presented above that the drift current may be completely determined from these three parameters. True, we have introduced two as yet unknown quantities, k and σ, but they are real constants and should be determined empirically once and for all. In a stratified ocean the stability will evidently affect the drift current by introducing changes in k and possibly in σ. It would therefore seem that we have succeeded in separating the influence of wind (stress) and stability and thus created at least a skeleton theory of oceanic turbulence.

We shall now attempt a preliminary, rough estimate of the constant k in order to be able to express our results numerically.

For this purpose we accept the second solution, based on the assumption that the surface drift forms an angle of 45° with the stress. We assume that Ekman's semi-empirical formula (111) reasonably well expresses the relation between h and w_s in middle latitudes where observations are most abundant.

From (103) we find

$$h = \frac{5}{\sqrt{2}} \frac{k^2}{f} \cdot w_s. \tag{112}$$

Now assume that k has the value 0.12. At latitude 43° one finds

$$f = 2\Omega \sin L = 10^{-4} \tag{113}$$

and therefore, at this particular latitude,

$$h = 509\ w_s, \tag{114}$$

in fair agreement with Ekman's empirical formula. As a check we shall calculate the ratio between surface wind and surface drift for the same latitude, making use of the empirical relation (109). We set the density of sea water equal to one and find, from (103) and (109),

$$\frac{w_s}{W_o} = 1.33.10^{-2}. \tag{115}$$

According to Ekman's theory[16] the same ratio should have the value

$$\frac{w_s}{W_o} = \frac{\lambda}{\sqrt{\sin L}}, \tag{116}$$

where λ is a constant. This constant has been determined repeatedly and values have been obtained ranging from 0.0103 to 0.0160.[17] Assuming a mean value $\lambda = 0.0135$ and inserting the proper value of L (about 43°) in (116) we find

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[15] Later various other assumptions regarding A have been made and the corresponding velocity distributions calculated. Thus Ekman has investigated the case of a stress proportional to the square of the rate of shear, and H. Solberg has treated the case of an eddy-viscosity coefficient of the form $a(z+b)^2$. See H. Solberg: Sur le frottement dans les couches basses de l'atmosphère, *Förh. Skand. Naturforskarmötet i Göteborg*, 57, p. 95.

[16] A. Defant: l. c. fn. 14, p. 74.

[17] A recent determination from current measurements on Finnish lightships has been made by E.Palmén: Zur Bestimmung des Triftstromes aus Terminbeobachtungen, *Journal du Conseil International pour l'Exploration de la Mer*, Vol. VI, No. 3, 1931.

(117) $$\frac{1}{\sqrt{\sin L}} = 1.2$$

and thus, from Ekman's theory, for this particular latitude

(118) $$\frac{w_s}{W_o} = 1.62.10^{-2},$$

or somewhat in excess of the value found from the present theory (115). At the pole the Ekman ratio would have the value $1.35.10^{-2}$. It should be emphasized that this ratio according to Ekman depends upon the latitude, since the surface drift produced by a given stress varies with latitude. The theory presented above implies that the drift is a function of the stress but not of the latitude. The influence of latitude will be noticed first when we consider the value of the mixing length next to the surface.

To determine the mixing length we shall make use of (IV), (73) and (70). The result is

(119a) $$l = k\frac{C'}{C''} = k\left|\frac{dz}{d\psi}\right| = k\frac{q\cos\sigma}{3}(h-z) \begin{cases} \dfrac{k(h-z)}{\sqrt{2}} & (\sigma = 0) \\[2ex] \dfrac{k(h-z)}{2} & (\sigma = 18°\ 26') \end{cases}$$

Thus l has a maximum at the surface and decreases in a linear fashion with depth; it disappears at the lower boundary of the drift current. Since l measures the vertical displacements due to the eddying motion within the drift current, it is evident that its value must decrease with increasing stability. The stability is zero when the wind velocity reaches such a value that the stirring of the ocean water becomes complete. We may therefore expect to observe mixing lengths comparable to those indicated by (119) in strong winds; in light winds the observed values of l must be less. At the level $z = o$ the expression for l reduces to

(119b) $$l_o = \frac{kh}{\sqrt{2}} = \frac{3k^2}{f\sqrt{2}}\sqrt{\frac{\tau_o}{\rho}} = 305\cdot\frac{10^{-4}}{f}\sqrt{\frac{\tau_o}{\rho}} \qquad (\sigma = 0)$$

It is seen that the maximum mixing length increases towards the equator as $\dfrac{1}{\sin L}$.

Knowing l we are in a position to calculate the eddy-viscosity coefficient, and find

(120a) $$A = \rho l^2 C' \begin{cases} \rho\dfrac{f(h-z)^2}{3\sqrt{2}} & (\sigma = 0) \\[2ex] \rho\dfrac{f(h-z)^2}{2\sqrt{10}} & (\sigma = 18°\ 26') \end{cases}$$

Thus the viscosity rapidly decreases with increasing depth, a result which agrees well with observations by Sverdrup[18] and others. The maximum value of A occurs at the level $z = 0$. There we find, with the aid of the expressions (92) and (103) for h,

[18] H. U. Sverdrup: The Wind-Drift of the Ice on the North-Siberian Shelf, *The Norwegian North Polar Expedition with the Maud*, Scientific Results, Vol. 4, No. 1, p. 27. See also E. Palmén, l. c. fn. 17, p. 397.

(120b)
$$A_{\max} \begin{cases} \sqrt{4.5}\cdot k^2 \dfrac{\tau_0}{f} = 9.8\cdot\dfrac{10^{-8}}{f} W_o^2 & (\sigma = 0) \\ \sqrt{2.5}\cdot k^2 \dfrac{\tau_0}{f} = 7.3\cdot\dfrac{10^{-8}}{f} W_o^2 & (\sigma = 18^\circ\ 26') \end{cases}$$

Equation (120b) is not correct dimensionally, since it was obtained by means of Ekman's formula (109), in which the air density is suppressed. The maximum values obtained in (120b) for middle latitudes ($f = 10^{-4}$) are about twice as large as certain characteristic mean values for the whole turbulent layer given by W. Schmidt.

The preceding theory leads to surface values of the mixing length and of the eddy viscosity which become infinite with approach to the equator. In reality the surface values of these quantities depend upon the immediate interaction between air and water and must therefore be functions of τ_o, the wind stress, and of ϵ, the vertical measure of the roughness of the sea surface. In a permanent wind blowing over a large ocean expanse, ϵ is a function of τ_o. Thus the surface values of the mixing length and of the eddy viscosity depend upon τ_o alone.

We must therefore assume that there exists, next to the ocean surface, a certain transition layer within which l increases from its fixed surface value l_o to the theoretically determined maximum value (119b). It is furthermore assumed that l_o and ϵ are identical. It is evident from the equation (119b) that the transition layer must increase in depth with decreasing latitude. However, excluding a belt next to the equator, we shall assume the transition layer to be everywhere so thin that within it variations of the stress with depth may be disregarded. This implies that all mass or volume forces (deflecting force, horizontal pressure gradients) are neglected.

Within the same layer it is assumed that l increases linearly with depth, ($l = k_o z + \epsilon$). To the coefficient k_o we assign the value 0.38, obtained by v. Kármán for two-dimensional motion. The z-coordinate is now counted from the ocean surface and has the value H at the bottom of the transition layer.

Using Prandtl's original expression for the stress (1) and remembering that the stress is assumed to be constant and directed along the x-axis, we find

(121)
$$\frac{\tau_o}{\rho} = k_o^2 \left(z + \frac{\epsilon}{k_o}\right)^2 \left(\frac{du}{dz}\right)^2,$$

or, after integration,

(122a)
$$u_s = u_o - \frac{1}{k_o}\sqrt{\frac{\tau_o}{\rho}}\ \log \frac{k_o H + \epsilon}{\epsilon}.$$

In (122a) u_o and u_s represent the components of the velocity along the x-axis (direction of the wind stress). Between u_s and w_s, the total velocity at H, exists the relation

(122b)
$$u_s = \frac{w_s}{\sqrt{3}} = 0.577\ w_s. \qquad (\sigma = 0.)$$

In view of our previous assumptions the y-component of the velocity remains constant throughout the transition layer and equals

(122c)
$$v = w_s \sqrt{\frac{2}{3}} = 0.816\ w_s.$$

The mixing length must be continuous at $z = H$. Thus

$$k_o H + \epsilon = \frac{kh}{\sqrt{2}} = \frac{3k^2}{f\sqrt{2}} \sqrt{\frac{\tau_o}{\rho}}\,. \tag{123}$$

From (122), (123) and (122c) we obtain

$$w_o \cos \varphi_o = \frac{w_s}{\sqrt{3}} + \frac{1}{k_o} \sqrt{\frac{\tau_o}{\rho}} \log \left(\frac{3k^2}{f \epsilon \sqrt{2}} \sqrt{\frac{\tau_o}{\rho}} \right) \tag{124a}$$

$$w_o \sin \varphi_o = w_s \sqrt{\frac{2}{3}} \tag{124b}$$

φ_o represents the angle between the stress and the surface drift (w_o). This result shows that the surface drift must reach very high values at low latitudes.

In order to estimate numerically w_o and φ_o we make use of (90) and (124).

$$w_o \cos \varphi_o = \frac{w_s}{\sqrt{3}} \left[1 + \frac{3k}{\sqrt{2}\, k_o} \log \frac{3k^2}{f \epsilon \sqrt{2}} \sqrt{\frac{\tau_o}{\rho}} \right], \tag{125a}$$

$$w_o \sin \varphi_o = w_s \sqrt{\frac{2}{3}}\,. \tag{125b}$$

According to (109) we have,

$$\sqrt{\tau_o} = 0.18 \cdot 10^{-2}\, W_o \qquad (W_o = \text{surface wind}). \tag{126}$$

Cornish[19] has given a rule, according to which the maximum roughness of the ocean surface is proportional to the wind velocity and given by

$$\epsilon = 0.37\, W_o. \tag{127}$$

Krümmel finds that this formula gives too high ϵ-values for light winds and too small ϵ-values for winds between 15 and 30 m.p.s. It may however be accepted as reasonably accurate for our purposes. Assuming $k = 0.12$ and making use of (126) and (127) we are in a position to evaluate (125) numerically. One finds

$$\tan \varphi_o = \frac{\sqrt{2}}{1 + 1.54\ {}^{10}\!\log \dfrac{1.5}{10^4 f}} \tag{128}$$

$$\frac{w_o}{w_s} = \sqrt{1 + 1.03\ {}^{10}\!\log \frac{1.5}{10^4 f} + 0.79 \left({}^{10}\!\log \frac{1.5}{10^4 f} \right)^2} \tag{129}$$

From (90), (126) and (129) we find, for the relation between surface drift and surface wind (W_o)

$$\frac{w_o}{W_o} = 1.225.10^{-2} \sqrt{1 + 1.03\ {}^{10}\!\log \frac{1.5}{10^4 f} + 0.79 \left({}^{10}\!\log \frac{1.5}{10^4 f} \right)^2} \tag{130}$$

[19] Krümmel: l. c. fn. 12, Band II, p. 74.

To obtain the depth of the transition layer it is necessary to go back to (123), (126) and (127). The result is

$$\frac{H}{h} = 0.223\,[1 - 0.673.10^4 f] \tag{131}$$

and therefore, for the total depth of the drift current,

$$H + h = \frac{7.93}{10^4 f}[1 - 0.123.10^4 f]\,W_o \tag{132}$$

In Table 1 we have calculated φ_o, $\frac{w_o}{w_s}$, $\frac{w_o}{W_o}$, $\frac{H}{h}$ and $(H+h)$ as functions of the latitude. $H+h$ was computed for a wind velocity of 10 m.p.s. For the sake of comparison we have also included two additional columns. One of them represents the ratio $\frac{w_o}{W_o}$ as calculated from Ekman's formula

$$\frac{w_o}{W_o} = \frac{\lambda}{\sqrt{\sin L}} \qquad (\lambda = 0.0135,\ L = \text{latitude}). \tag{133}$$

The last column gives D, the "depth of frictional influence," as a function of latitude. It is given by

$$D = \frac{7.5\,W_o}{\sqrt{\sin L}}.\text{[20]} \tag{134}$$

The variation of $(H+h)$ with latitude agrees fairly well with observations. Defant quotes values for the depth of the drift current of 50 m. in the Weddell Sea (lat. 65° S) and 150 m. in lat. 8° N. It is evident that the present formula leads to a more rapid variation with latitude than does Ekman's theory.

TABLE 1

Calculated characteristics of drift current maintained by a steady surface wind of 10 m.p.s.

L	φ_o	$\frac{w_o}{w_s}$	$\frac{w_o}{W_o}10^2$	$\frac{H}{h}$	$H+h$ Meters	$\left(\frac{w_o}{W_o}10^2\right)_{Ekman}$	D Meters
15°	36.3	1.38	1.69	0.166	201	2.65	147
30°	43.6	1.18	1.45	0.114	99	1.91	106
45°	48.5	1.09	1.34	0.068	67	1.61	89
60°	51.8	1.04	1.27	0.034	53	1.45	81
75°	53.6	1.01	1.24	0.012	47	1.37	76
90°	54.2	1.01	1.23	0.004	45	1.35	75

[20] Krümmel: l. c. fn. 12, Band II, p. 545.

V. APPLICATION TO THE ATMOSPHERE

Place the origin at the surface of the earth and let the z-axis point upward. The y-axis is directed 90° to the left of the x-axis. Then the horizontal deflecting force per unit volume is

$$D_x + iD_y = \rho f e^{-\frac{\pi}{2}i}(U+iV). \tag{135}$$

The equations of motion may be compressed in the form

$$\rho f e^{-\frac{\pi}{2}i}(U+iV) - \frac{\partial p}{\partial x} - i\frac{\partial p}{\partial y} + \frac{\partial}{\partial z}\left[\rho k^2 K^2\left(\frac{dz}{d\psi}\right)^2 e^{i(\psi+\sigma)}\right] = 0. \tag{136}$$

The horizontal pressure gradient is constant. At high levels gradient wind is reached. We may rotate the coordinate system in such a way as to make the x-axis coincide with the direction in which the gradient wind is blowing. In that case

$$\rho f e^{-\frac{\pi}{2}i} U_g - \frac{\partial p}{\partial x} - i\frac{\partial p}{\partial y} = 0 \tag{137}$$

and, therefore, at any level,

$$\rho f e^{-\frac{\pi}{2}i}\left[U+iV-U_g\right] + \frac{\partial}{\partial z}\left[\rho k^2 K^2\left(\frac{dz}{d\psi}\right)^2 e^{i(\psi+\sigma)}\right] = 0. \tag{138}$$

To solve this equation we may proceed as in the case of drift currents. Differentiate with respect to z and introduce

$$\frac{d}{dz}(U+iV) = Ke^{i\psi}. \tag{139}$$

The result is

$$\frac{f}{k^2K}e^{i\psi-(\frac{\pi}{2}+\sigma)i} + \frac{d^2}{dz^2}\left[\left(\frac{dz}{d\psi}\right)^2 e^{i\psi}\right] = 0. \tag{140}$$

With the above procedure an integration constant is introduced which must be determined through substitution of the solution of (140) in the original equation (138). From (140) one obtains

$$2z''' + 3iz'' - z' + q\cdot e^{-(\frac{\pi}{2}+\sigma)i}\cdot z' = 0. \qquad \left(q = \frac{f}{k^2K}\right) \tag{141}$$

Thus

$$2z''' = z'\left[1+q\sin\sigma\right] \tag{142a}$$

$$3z'' = z' q \cos\sigma. \tag{142b}$$

A necessary and sufficient requirement for the simultaneous fulfillment of these two equations is that

$$\frac{q^2\cos^2\sigma}{9} = \frac{1+q\sin\sigma}{2} \tag{143}$$

We find, in the same way as before,

$$z = h\left[1 - e^{\frac{q\cos\sigma}{3}(\psi - \psi_s)}\right] \qquad (\psi_{z=0} = \psi_s,\ \psi_{z=h} = -\infty) \tag{144}$$

$$\frac{dz}{d\psi} = -h\frac{q\cos\sigma}{3}\cdot e^{\frac{q\cos\sigma}{3}(\psi - \psi_s)} \tag{145}$$

To determine the velocity $U + iV$ we make use of the fact that

$$U + iV = U_s + iV_s + \int_0^z K e^{i\psi} dz \qquad (U_s + iV_s = \text{surface wind}) \tag{146}$$

or

$$U + iV = U_s + iV_s - Kh\frac{q\cos\sigma}{3}e^{-\frac{q\cos\sigma}{3}\psi_s}\int_{\psi_0}^{\psi} e^{\psi\left(i + \frac{q\cos\sigma}{3}\right)} d\psi. \tag{147}$$

If the integration is carried out one obtains

$$U + iV = U_s + iV_s - Kh\frac{q\cos\sigma}{q\cos\sigma + 3i}e^{-\frac{q\cos\sigma}{3}\psi_s}\left[e^{\psi\left(i + \frac{q\cos\sigma}{3}\right)} - e^{\psi_s\left(i + \frac{q\cos\sigma}{3}\right)}\right] \tag{148}$$

This equation is simplified through the introduction of an angle β defined by

$$\operatorname{cotg}\beta = \frac{q\cos\sigma}{3}. \tag{149}$$

Then

$$U + iV = U_s + iV_s - Kh\cdot\cos\beta\cdot e^{-i\beta - \operatorname{cotg}\beta\cdot\psi_s}\left[e^{\psi(i + \operatorname{cotg}\beta)} - e^{\psi_s(i + \operatorname{cotg}\beta)}\right]. \tag{150}$$

Taking advantage of the fact that $U + iV$ reaches its gradient value at the level h, we find

$$U_g = U_s + iV_s + Kh\cos\beta\cdot e^{i(\psi_s - \beta)} \tag{151}$$

With the aid of this expression we may eliminate K and h from (150) and obtain finally

$$U + iV = U_g - \left[U_g - (U_s + iV_s)\right] e^{(i + \operatorname{cotg}\beta)(\psi - \psi_s)} \tag{152}$$

Now let us go back to the original equation of motion (138). After substitution of the expression for $\frac{dz}{d\psi}$ and completion of the differentiation we find

$$\rho f e^{-\frac{\pi}{2}i}[U + iV - U_g] - \rho k^2 K^2 e^{i(\psi + \sigma) + \operatorname{cotg}\beta(\psi - \psi_s)}[2h\operatorname{cotg}^2\beta + ih\operatorname{cotg}\beta] = 0. \tag{153}$$

Thus, at the surface,

$$\rho f e^{-\left(\frac{\pi}{2} + \psi_s + \sigma\right)i}[U_s + iV_s - U_g] = \rho k^2 K^2 h\operatorname{cotg}\beta\,[2\operatorname{cotg}\beta + i]. \tag{154}$$

We may now insert the proper value for $U_s + iV_s$ from (151) and obtain

$$-\rho K \cos\beta\, e^{-(\frac{\pi}{2}+\beta+\sigma)i} = \rho k^2 K^2 h \operatorname{cotg}\beta\,[2\operatorname{cotg}\beta + i]. \tag{155}$$

After some rearrangements this changes into

$$f \sin\beta\, e^{(\frac{\pi}{2}-\beta-\sigma)i} = k^2 K\,[2\operatorname{cotg}\beta + i] \tag{156}$$

Thus

$$\tan(\beta+\sigma) = 2\operatorname{cotg}\beta \tag{157}$$

and

$$q = \frac{f}{k^2 K} = \sqrt{\frac{4\operatorname{cotg}^2\beta + 1}{\sin^2\beta}} \tag{158}$$

The second of these equations does not give anything new since it may be obtained through the elimination of σ between (149) and (157).

In order to complete the determination of the wind we have to impose certain conditions at the lower boundary. It is obvious that the solution just derived cannot apply to the layers in the immediate vicinity of the ground. We shall therefore, restrict our solution to the region between $z = h$ and a level, as yet undetermined, next to the ground where the preceding solution loses its validity. We assume the origin of the coordinate system to be located at this particular level and understand by $U_s + iV_s$ the velocity vector prevailing there. It is natural to demand that wind and stress at this level be parallel. The stress may be calculated from VII, (145) and (149):

$$(\tau_x + i\tau_y)_{z=0} = \tau_s e^{i(\psi_s+\sigma)} = \rho k^2 K^2 h^2 \operatorname{cotg}^2\beta\, e^{i(\psi_s+\sigma)} \tag{159}$$

Thus

$$U_s + iV_s = W_s e^{i\varphi_s} = W_s e^{i(\psi_s+\sigma)} \qquad (\varphi_s = \text{direction of surface wind}) \tag{160}$$

Inserting this relation in (151) one finds

$$W_s = U_g e^{-i(\psi_s+\sigma)} - Kh\cos\beta\, e^{-i(\beta+\sigma)} \tag{161}$$

or, after separation of real and imaginary parts,

$$W_s = U_g[\cos\varphi_s - \operatorname{cotg}(\beta+\sigma)\sin\varphi_s] = U_g \frac{\sin(\beta+\sigma-\varphi_s)}{\sin(\beta+\sigma)} \tag{162a}$$

$$Kh\cos\beta = \frac{U_g \sin\varphi_s}{\sin(\beta+\sigma)} \tag{162b}$$

With the aid of the last equation the expression for the stress becomes

$$\tau_s = \rho k^2 U_g^2 \frac{\sin^2\varphi_s}{\sin^2(\beta+\sigma)\sin^2\beta} = \rho k^2 W_s^2 \frac{\sin^2\varphi_s}{\sin^2\beta\,\sin^2(\beta+\sigma-\varphi_s)} \tag{163}$$

We have now brought our analysis to a point where it is possible to determine the wind distribution in all its details, provided we know the constants k, σ (which should be determined once and for all), the external parameters U_g, ρ and f, which vary from case to case, and one of W_s or φ_s.

The natural assumption with regard to σ is that stress and shearing vector are parallel, that is, $\sigma = 0$. In this case it follows from (157) that

(164) $$\tan \beta = \sqrt{2}, \quad \beta = 54° \, 44'$$

and from (149) and (158)

(165) $$q = \frac{3}{\sqrt{2}}, \; K = \frac{f\sqrt{2}}{3k^2}$$

Under those circumstances (161), (162) and (163) may be reduced to

(166) $$W_s = U_g \left(\cos \varphi_s - \frac{1}{\sqrt{2}} \sin \varphi_s \right) = U_g \sqrt{\frac{3}{2}} \sin (\beta - \varphi_s)$$

(167) $$h = \frac{9}{2} \cdot \frac{k^2}{f} \cdot U_g \sin \varphi_s$$

(168) $$\tau_s = \frac{9}{4} \rho k^2 U_g^2 \sin^2 \varphi_s = \frac{3}{2} \rho k^2 W_s^2 \frac{\sin^2 \varphi_s}{\sin^2 (\beta - \varphi_s)} .$$

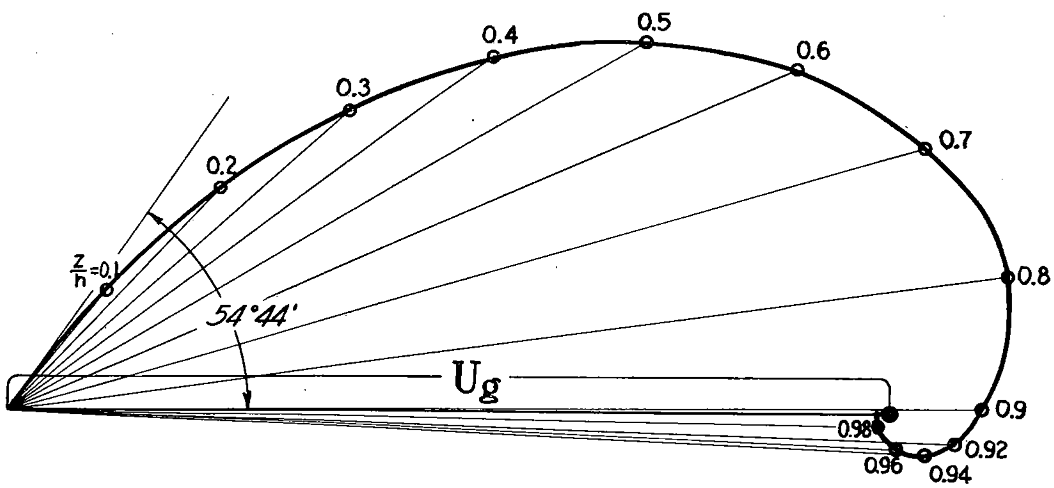

FIG. 2

The mixing length l is given by

(169) $$l = k \left| \frac{dz}{d\psi} \right| = \frac{k (h - z)}{\sqrt{2}}; \quad l_s = \frac{kh}{\sqrt{2}} = \frac{9}{2\sqrt{2}} \frac{k^3}{f} U_g \sin \varphi_s$$

The turbulent layer reaches its maximum height when φ_s has its maximum value (54° 44′), that is, when $W_s = 0$. Then

(170) $$h_{\max} = \frac{3\sqrt{3}}{2} \frac{k^2}{f} U_g$$

and

(171) $$\tau_{s\,\max} = \frac{3}{4} \rho k^2 U_g^2$$

The wind distribution corresponding to this particular value of φ_s is represented graphically in fig. 2.

G. I. Taylor[21] has found that the relation between surface wind and gradient wind conforms, to a high degree of approximation, with the equation

$$W_s = U_g\,(\cos\varphi_s - \sin\varphi_s). \tag{172}$$

It is as yet uncertain if we may identify the surface wind with the quantity W_s in (162a), but if that is the case, then we are forced to accept another solution, namely

$$\operatorname{cotg}(\beta+\sigma) = 1; \quad \beta+\sigma = \frac{\pi}{4}. \tag{173}$$

It follows from (157) that

$$\tan\beta = 2, \quad \beta = 63^\circ\,26', \quad \sigma = -18^\circ\,26' \tag{174}$$

and from (158) that

$$q = \frac{f}{k^2 K} = \frac{1}{2}\sqrt{10}. \tag{175}$$

By substitution of (173) and (174) into (163) we obtain

$$\tau_s = \frac{5}{2}\rho k^2 U_g^2 \sin^2\varphi_s = \frac{5}{4}\rho\, k^2\, W_s^2 \frac{\sin^2\varphi_s}{\sin^2\left(\frac{\pi}{4}-\varphi_s\right)}. \tag{176}$$

From (162b) follows

$$h = 5\frac{k^2}{f}\,U_g \sin\varphi_s, \quad l = \frac{k\,(h-z)}{2}, \quad l_s = \frac{kh}{2} = \frac{5}{2}\,\frac{k^3}{f}\,U_g \sin\varphi_s. \tag{177}$$

The two alternative solutions presented above show that the mixing length has its maximum value at the lower boundary, $z = 0$. On the other hand, for physical reasons, the mixing length next to a rigid wall must have a prescribed value which is equal to zero if the surface is smooth. It is therefore clear that the solutions obtained cannot apply all the way to the ground. One is therefore led to conclude that there exists, also in the atmosphere, a certain transition layer within which the mixing length gradually approaches the theoretical value prescribed by (169) or (177).

It is easy to show that this layer, in view of the continuity of the mixing length, must become deeper as one approaches the equator. A satisfactory theory for the motion within the transition layer must enable one to calculate the angle φ_s which the preceding theory left undetermined. Since φ_s is a pure number, it must be a function of the non-dimensional quantities which may be formed from the controlling external parameters, namely $\frac{U_g\epsilon}{\nu}$ and $\frac{\epsilon f}{U_g}$. The first is a Reynolds' number obtained from the wind, the characteristic linear measure ϵ of the roughness of the surface and the kinematic viscosity of air. It is easy to show, with the aid of available observational material, that φ_s must be a function of *both* of these quantities.

It has not been possible for me to develop a satisfactory theory of this type. The following application of the theoretical results will therefore be restricted to conditions in the free atmosphere.

[21] G. I. Taylor: l. c. in fn. 5.

G. I. Taylor[22] has derived and verified from observations at Salisbury Plain, a formula for the surface stress:

$$\tau = \rho \kappa W^2 \qquad (\kappa = \text{constant}, \qquad W = \text{wind velocity}) \tag{178}$$

The wind was measured at an elevation of 30 meters above the ground. From an estimate of the height of the transition layer given below, it appears probable that W must agree fairly closely with the wind velocity W_s occurring in (168) and (176). Thus from a comparison of (168) and (178) it follows that:

$$\kappa = \frac{3}{2} k^2 \frac{\sin^2 \varphi_s}{\sin^2 (\beta - \varphi_s)} \tag{179}$$

Taylor cites three determinations of κ:

(a) light winds, $\varphi_s = 13°$, $\kappa = 0.0023$
(b) moderate winds, $\varphi_s = 21.5°$, $\kappa = 0.0032$
(c) strong winds, $\varphi_s = 20°$, $\kappa = 0.0022$

From these data one obtains:

(a) $k = 11.6 \cdot 10^{-2}$ (b) $k = 6.9 \cdot 10^{-2}$ (c) $k = 6.4 \cdot 10^{-2}$

The first of these determinations, although in good agreement with the value obtained for ocean drift currents, must in all likelihood be disregarded, since the low wind velocity and the small value of the angle φ_s indicate incomplete stirring of the atmosphere. Furthermore, in case of very light winds, local horizontal pressure gradients are apt to make the angle φ_s and thus the value of k quite unreliable. We shall therefore accept the two last determinations only and introduce the value $k = 6.5 \cdot 10^{-2}$ in our subsequent determinations.

As a check it is of interest to calculate K, the rate of shear of the mean motion. This quantity is given by (165). We find for middle latitudes ($f = 10^{-4}$), $K = 1.12 \cdot 10^{-2}$. In southern England (lat. 50°), the corresponding value would be about $1.25 \cdot 10^{-2}$, meaning an increase in wind velocity of 1.25 mps. per hundred meters of elevation. This agrees well with certain data collected by J. S. Dines and reproduced by Shaw.[23] It should be remembered that the above value of $1.25 \cdot 10^{-2}$ does not apply to the transition layer next to the ground where the rate of increase of the wind velocity with height is much more rapid.

We may now tabulate the height h of the turbulent layer from (169). The result is presented in Table 2, for various latitudes and various values of the angle φ_s. The gradient wind was assumed to be 15 mps. The theoretical values obtained are of the same order of magnitude as those observed, but since no systematic study has been made of the actual variations in height of the turbulent layer with wind velocity, latitude and orientation of the surface wind, it is impossible to carry out a detailed comparison. However, an inspection of the data collected by J. S. Dines and referred to above, would seem to indicate that the height of the turbulent layer rapidly increases with the gradient wind velocity, as indicated by the present theory. 167 ?

It is evident that the depth of the transition layer next to the ground must at least

[22] G. I. Taylor: Skin Friction of the Wind on the Earth's Surface, *Proceedings of the Royal Society*, Series A, Vol. 92, No. A637, 1916.

[23] Sir Napier Shaw: Manual of Meteorology, Volume IV (Revised edition of Part IV), *Cambridge University Press*, 1931, fig. 23, p. 124.

equal the theoretical maximum value of the mixing length obtained from the present theory. This maximum value increases towards the equator at the rate $\frac{1}{f}$ and one must therefore assume that the transition layer also increases in depth at the same rate. Under reasonable assumptions with regard to U_g and φ_s and excluding the equatorial belt, $l_{z=0}$ will vary between 25 meters and 100 meters. These values should therefore represent the minimum thickness of the transition layer.

The "Austausch" coefficient at the lower boundary of the free atmosphere is obtained from the expression:

$$A = \rho \frac{k^2 h^2}{2} K = \rho \frac{h^2 f}{3\sqrt{2}} = 10^{-7} \frac{U_g^2}{f} \sin^2 \varphi_s. \tag{180}$$

This formula gives the maximum value of A in a vertical column for different values of the gradient wind velocity and at different latitudes. Above the level $z = 0$ where this value prevails, A decreases as $(h - z)^2$. This is in good agreement with a theoretical result obtained by the writer on a previous occasion.[24] Convective phenomena, the presence of water vapor in the atmosphere and variations in the direction and magnitude of the horizontal pressure gradient with altitude tend to obscure this last result for which, however, Sverdrup[25] and Möller[26] seem to have found some support. The numerical coefficient in the right member of (180) is proportional to the fourth power of k. A few determinations not quoted above would seem to indicate that the latter constant is closer to 0.055 than to 0.065. This has the effect of reducing the coefficient in (180) to about $0.5 \cdot 10^{-7}$.

One may use the data in Table 2 to calculate the mixing length from (169) or (177). The numerical values obtained are somewhat smaller than the linear dimensions of turbulence elements determined by K. O. Lange[27] from observations on floating pilot balloons. It should however be remembered that the mixing length as here defined depends upon the correlation between the vertical and horizontal turbulent velocity components and therefore cannot be compared directly with measured turbulence elements.

TABLE 2

Height, in meters, of turbulent layer for a gradient wind of 15 mps.

L \ φ_s	10°	15°	20°	25°	30°	35°	40°	45°
15°	1313	1958	2587	3197	3783	4339	5619	5349
30°	679	1012	1338	1653	1956	2244	2515	2766
45°	480	716	946	1169	1383	1587	1778	1956
60°	392	584	772	954	1129	1295	1451	1597
75°	351	524	692	855	1012	1161	1301	1431
90°	340	506	669	827	978	1122	1257	1383

24 C.-G. Rossby: l. c. in fn. 8.

25 H. U. Sverdrup: Wärmehaushalt und Austauschgrösse auf Grund der Beobachtungen der „Maud"-Expedition, *Beiträge zur Physik der freien Atmosphäre*, Bjerknes-Festband (XIX. Band), p. 276.

26 F. Möller: l. c. in fn. 4.

27 K. O. Lange: Über Windströmungen an Hügelhindernissen, *Veröffentlichung des Forschungs-Institutes der Rhön-Rossitten-Gesellschaft*, No. 4, Munich, 1931.